TIME SHIFTING CHAOTIC SIGNALS USING SYNCHRONIZATION

Jonathan N. Blakely and Ned J. Corron
U. S. Army Aviation and Missile Research, Development and Engineering Center
AMSRD-AMR-WS-ST, Redstone Arsenal, Alabama 35898, USA

ABSTRACT

In this paper we examine two couplings that produce time-shifted synchronization in a pair of chaotic oscillators. The couplings do not utilize an explicit time-delay term. We characterize the observed synchronization by determining the dependence of the time shift and cross correlation between the drive and response oscillators on a tunable parameter. Our observations agree well with estimates of the time shift and cross correlation using a transfer function.

1. INTRODUCTION

Coupled chaotic oscillators often display some form of synchronous behavior [1, 2]. Recently, varieties of synchronization have been identified where the oscillator waveforms are nearly but not exactly identical [3-9]. In these cases, either the coupling or a parameter mismatch produces a time-shifted synchronization state with some finite amplitude error (e. g., see Fig. 1). It remains an interesting open question whether natural systems utilize such mechanisms. On the other hand, in engineered systems applications have been identified where an easily varied time delay is desired.

Recently, chaotic oscillators have been suggested as efficient sources for generating wide-bandwidth waveforms for various applications. For example, the broadband and non-repeating nature of chaos provides an ideal combination of high range resolution and no range ambiguity for radar. Local coupling can synchronize an extended array of chaotic oscillators, thereby providing a coherent state suitable for power combining in a wide-bandwidth phased array antenna. Moreover, time-shifted synchronization provides a mechanism to steer the radiated beam. A tunable coupling strategy may be used that continuously varies the time shift, thereby electronically steering the beam to a desired direction [3, 5, 6, 9]. Effectively, these systems trade some decrease in cross-correlation between oscillators for a practical means of obtaining a tunable time shift. In practice, this tradeoff appears as reduced beam quality for larger steering angles and limits the maximum steering angle for these arrays.

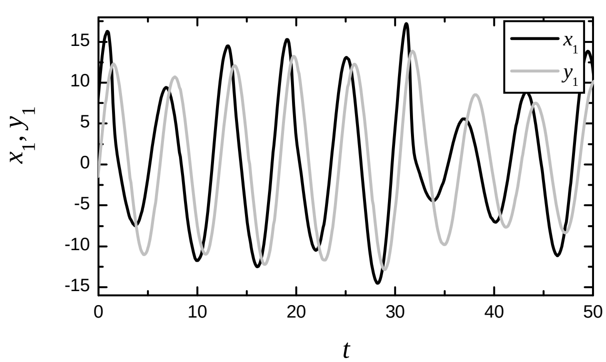

Figure 1. Example of time-shifted synchronization in a pair of coupled Rössler oscillators. The response waveform $y_1(t)$ lags the drive waveform $x_1(t)$ and is slightly attenuated.

In this paper, we examine two unidirectional coupling schemes that produce non-trivial time shifts and cross-correlations, and we compare these observations to predictions based on a new method for estimating characteristics of the synchronized state [10]. This method uses a transfer function derived from the coupling model to provide estimates of the time shift and amplification or attenuation between the drive and response waveforms. Each coupling scheme involves a parameter to tune the time shift, and typically a large range of time shifts can be obtained. Overall, we find that observed time shifts and cross-correlations agree well with estimates based on the transfer function and that the agreement increases with coupling strength.

2. TIME-SHIFTED SYNCHRONIZATION

We consider a pair of identical chaotic oscillators coupled unidirectionally of the form

$$\dot{x} = f(x), \quad \dot{y} = f(y) + g(x, y) \tag{1}$$

where $x(t)$ and $y(t)$ are the state vectors of the drive and response oscillator, respectively, f is a chaotic flow, and g is a linear coupling function. Since f is chaotic, it is necessarily nonlinear and the state vectors x and y each contain at least three scalar states.

Here, we consider g of the form

$$g(x, y) = k\{Px_1 - Qy_1\}\hat{e}_1, \tag{2}$$

where x_1 and y_1 are the first components of the state vectors x and y, respectively, k is the coupling strength, $\hat{e}_1 = (1, \quad 0, \quad 0)^T$, and P and Q are linear scalar operators. At least five coupling schemes of this form have previously been identified that produce approximate time-

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shifted synchronization [6-8, 10 ,11]. In these cases, identical synchronization is not an exact solution of Eqs. (1) and (2). Instead, we consider the drive and response synchronized when

$$k\{Px_1 - Qy_1\} = \varepsilon(t) \tag{3}$$

and $\varepsilon(t)$ is small compared to the first component of the vector field f. In other words, the coupling term only provides a small perturbation to the natural dynamics of the response oscillator.

Apart from a time shift, the waveforms $x_1(t)$ and $y_1(t)$ are typically similar in shape, with the primary distortion being just a change in amplitude. Thus, we can write

$$y_1(t) \approx Ax_1(t-\tau) \tag{4}$$

where τ is the time shift and A is the approximate gain (or attenuation) of the response waveform relative to the drive. For a given lag δt, we define the cross correlation function

$$C(\delta t) \equiv \frac{\int x_1(t)\, y_1(t+\delta t)\, dt}{\int x_1(t)\, x_1(t)\, dt}, \tag{5}$$

where we note the normalization does not treat the drive and response waveforms symmetrically. Assuming Eq. (4), we recognize $\delta t = \tau$ maximizes the cross correlation and

$$C(\tau) \approx A \tag{6}$$

so that the cross correlation provides an estimate of the response gain.

It has been shown that a reasonable estimate of the time shift τ and cross correlation $C(\tau)$ can be derived by neglecting the small perturbation $\varepsilon(t)$ [10]. Thus, setting $\varepsilon(t) = 0$ in Eq. (3) and taking a Fourier transform, we can write

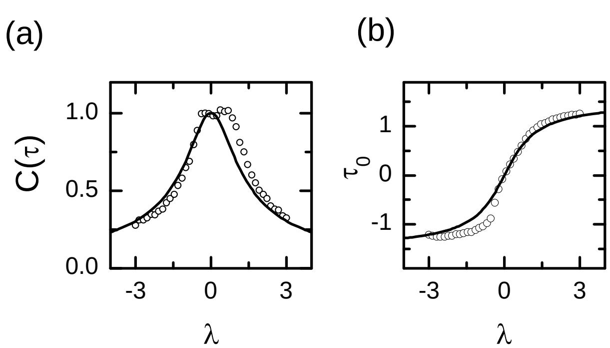

Figure 2. (a) Correlation and (b) time shift between coupled Rössler oscillators observed in numerical solutions (circles) and estimated using the transfer function Eq. (12) (line).

$$\hat{y}_1(\omega) = T(\omega)\,\hat{x}_1(\omega) \tag{7}$$

where $\hat{y}_1(\omega)$ and $\hat{x}_1(\omega)$ are the Fourier transforms of $x_1(t)$ and $y_1(t)$, respectively. We note that the transfer function $T(\omega)$ depends only on the linear operators P and Q in this approximation, and that for many simple operators $T(\omega)$ can be calculated analytically. The time shift between the drive and response is then predicted to be

$$\tau = \frac{\arg\left(T(\omega_0)\right)}{\omega_0} \tag{8}$$

while the predicted cross correlation is

$$C(\tau) = \left|T(\omega_0)\right| \tag{9}$$

where ω_0 is the dominant angular frequency of the specific chaotic oscillator under consideration, and $\arg\left(T(\omega_0)\right)$ represents the phase angle of the complex transfer function evaluated at this frequency.

Below we introduce two examples of couplings that produce approximate time shifted synchronization. Each coupling includes a tunable parameter that varies the time shift and cross correlation of the synchronized state. We determine the functional dependence of these quantities on the tunable parameter and compare numerical results with predictions made using a transfer function.

3. FIRST EXAMPLE

We first consider a pair of Rössler oscillators whose flow is determined by the vector field

$$f(x) = \begin{pmatrix} -x_2 - x_3 \\ x_1 + 0.15x_2 \\ 0.2 + (x_1 - 10)x_3 \end{pmatrix}, \tag{10}$$

and who are coupled by the function

$$g(x,y) = k\{x_1(t) - y_1(t) + \lambda \dot{y}_1(t)\}\hat{e}_1, \tag{11}$$

where $k = 0.3$ and λ is a tunable parameter. Typical time series of x_1 and y_1 with $\lambda = -0.5$, shown in Fig. 1, exhibit a state of time-shifted synchronization. The primary form of distortion of the response trajectory relative to the drive is a small degree of attenuation.

Following the procedure described in the preceeding section, we obtain the transfer function

$$T(\omega) \equiv \frac{\tilde{y}_1(\omega)}{\tilde{x}_1(\omega)} = \frac{1}{1 - i\omega\lambda}. \tag{12}$$

The spectral content of $x_1(t)$ is dominated by the single frequency $\omega_0 \approx 1.04$. The magnitude of the transfer function at ω_0, given by

$$\left|T\left(\omega_0\right)\right| = \frac{1}{\sqrt{1+\omega_0{}^2\lambda^2}}, \tag{13}$$

suggests that the amplitude of the response oscillation will be attenuated relative to the drive when $|\lambda| \neq 0$. The phase delay of the transfer function, divided by ω_0 predicts the time shift of the synchronized state. In this case,

$$\tau = \frac{\arg\left(T\left(\omega_0\right)\right)}{\omega_0} = \frac{1}{\omega_0}\tan^{-1}\left(\omega_0\lambda\right), \tag{14}$$

Notably, at large values of $|\lambda|$ the predicted time shift asymptotes to $\pm\pi/\left(2\omega_0\right)$.

We compare these theoretical estimates to observations of the correlation function, Eq. (5). The value of δt at which $C\left(\delta t\right)$ reaches its first maximum is identified as the observed time shift τ. The magnitude of $C\left(\delta t\right)$ at its maximum is a measure of the gain or attenuation of the response waveform relative to the drive. The time series data is generated by numerical solution of Eqs. (1), (10), and (11).

Fig. 2(a) shows the magnitude of the transfer function, $\left|T\left(\omega_0\right)\right|$ (solid line), agrees well with the observed correlation, $C\left(\tau\right)$ (circles), over a wide range of values of the parameter λ. Interestingly, the agreement is good even for large values of λ where $C\left(\tau\right) < 0.4$. This is surprising since, under these conditions, the approximation that $\varepsilon\left(t\right)$ is negligibly small is presumably rather poor.

The predicted values of the time shift also agree quite well with the observed time shift τ, as shown in Fig. 2(b). Most important, the observed asymptotic levels of the time shift agree with the values predicted by Eq. (14) to within a few percent. Again, it is surprising that the agreement is so good at large values of $|\lambda|$.

Thus far we have considered only a single value of the coupling strength, *k*, and have not addressed the role of coupling strength in determining the properties of the synchronized state. Obviously, the coupling strength must be sufficient to guarantee stability or the properties of the synchronized state will not be observable [1, 2]. However, there typically exists a range of coupling strengths over which synchronization is stable, and the question remains in this case whether the coupling strength further affects the time shift and cross correlation. One might guess that stronger coupling would tend to minimize the difference term $Px_1 - Qy_1$ in Eq. (1). Thus, the properties of the response state y_1 would more closely agree with the transfer function. Here we examine a particular example that is consistent with this hypothesis. (We note that the effects of increased coupling on the other states of the response may be quite different. In fact, the resemblance of y_2 and y_3 to their drive counterparts may actually decrease with increasing coupling strength.)

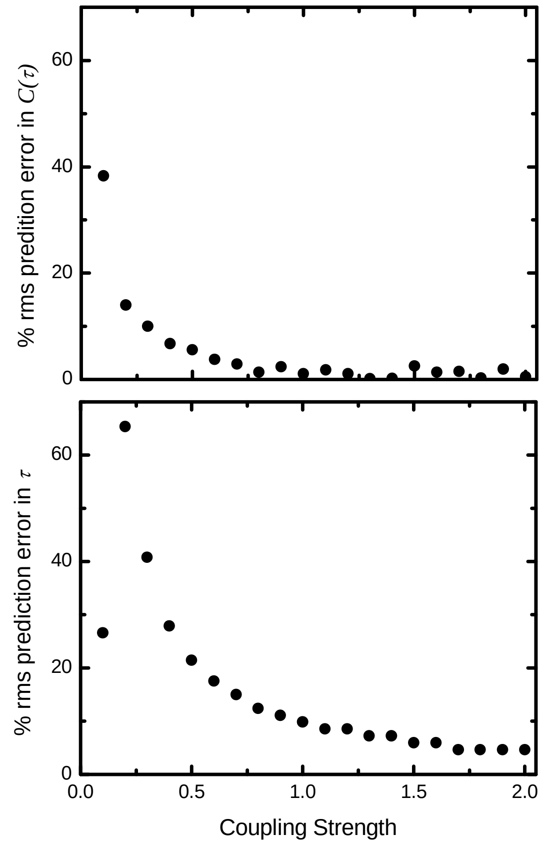

Figure 3. Root mean square error in the transfer function predictions of the (a) correlation and (b) time shift as a percentage of the observed value.

We focus on the range of coupling strengths from $k = 0.1$ to $k = 2.0$. The lower bound on this range is determined by the requirement that synchronization must be stable. We find that the transition between synchronized and unsynchronized dynamics occurs somewhere around $k = 0.1$.The upper bound is set by the fact that when $k = \lambda^{-1}$ the derivative term in the coupling exactly balances the derivative term on the left hand side of the equation. This cancellation changes the character of the equations and causes problems for numerical simulation. Thus we avoid it by constraining *k* to fall below λ^{-1} for the particular value of λ we wish to consider. We fix the value $\lambda = -0.5$ under the assumption that typical behavior occurs at this value.

Following the same procedures as described above, we can numerically integrate this system and observe the time shift and cross correlation as the coupling strength k is varied. We quantify the agreement between the observed values and the predictions of the transfer function by calculating the root mean square deviation between the observed and predicted values as a percent of

the observed value. Fig. 3 shows the prediction error in the (a) correlation and (b) time shift as functions of the coupling strength. As expected the prediction error decreases with increasing coupling strength over the entire range. Thus the transfer function provides a better prediction as the coupling strength is increased.

4. SECOND EXAMPLE

We next examine a variation of the previous coupling scheme that includes a second derivative. Specifically, the coupling scheme is

$$g(x,y) = k\left(x_1 - y_1 + \lambda \dot{y}_1 + \alpha \ddot{y}_1\right)\hat{e}_1, \tag{15}$$

where λ is a tunable parameter and α is a fixed parameter. The class of oscillators introduced by Sprott is one for which this coupling is easily implemented [12]. Here we use one such oscillator whose flow is defined by

$$f(x) = \begin{pmatrix} x_2 \\ x_3 \\ -0.6x_3 - x_2 + |x_1| - 2 \end{pmatrix}. \tag{16}$$

By numerical solution we determine the dominant frequency in the spectral content of $x_1(t)$ to be 0.958 ± 0.005. The transfer function for this coupling is

$$T(\omega) = 1 - \alpha\omega^2 + i\omega\lambda. \tag{17}$$

The predicted cross-correlation is

$$C(\tau) = \sqrt{(1-\alpha{\omega_0}^2)^2 + {\omega_0}^2\lambda^2}. \tag{18}$$

Notably, the correlation at $\lambda = 0$ is predicted to be $1-\alpha{\omega_0}^2$. All previous couplings that have been analyzed using the transfer function produced identical synchronization at $\lambda = 0$ [6-8, 10 ,11]. The predicted time shift is

$$\tau = \frac{1}{\omega_0}\tan^{-1}\left(\frac{\omega_0\lambda}{1-\alpha{\omega_0}^2}\right). \tag{19}$$

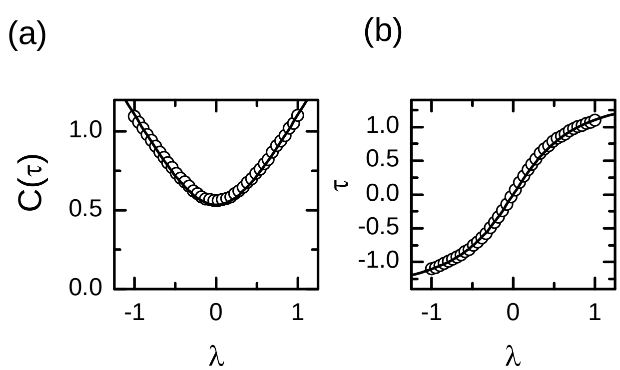

Figure 4. Observed (circles) and predicted (solid line) (a) cross-correlation and (b) time shift for the first coupling example.

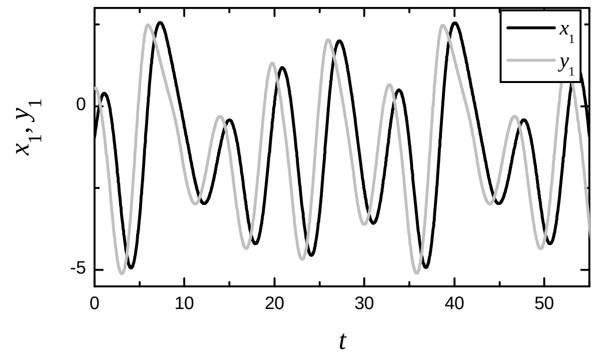

Figure 5. Example of time-shifted synchronization in a pair of coupled Sprott oscillators. The response waveform $y_1(t)$ anticipates the drive waveform $x_1(t)$.

We numerically integrate the system to generate time series of x_1 and y_1 at each of a range of values for λ and with $k = 20$ and $\alpha = 0.5$. From the numerical simulation, we estimate the time shift and attenuation from the cross-correlation function, Eq. (5). Fig. 4 shows the observed dependence (circles) of (a) $C(\tau)$ and (b) τ on the parameter λ. The observed time shift dependence on λ follows closely the prediction of Eq. (19). The observed cross-correlation at $\lambda = 0$ is 0.56 ± 0.005 which agrees well with the prediction of $1-\alpha{\omega_0}^2 = 0.541 \pm 0.005$. The correlation then grows quadratically as $|\lambda|$ increases, in agreement with Eq. (18) and reaches unity around $|\lambda| = 0.9$. Fig. 5 shows a typical time series of x_1 and y_1 with $\lambda = 0.9$ which verifies the absence of significant distortion despite the time shift.

5. CONCLUCIONS

We have shown two examples of couplings that produce time shifted synchronization and verified the accuracy of the transfer function in predicting the observed properties of the synchronized state. These results represent further evidence of both the wide range of coupling that can produce time-shifted synchronization and the usefulness of the transfer function in analyzing this phenomenon. In the first example, we observed that the accuracy improves with increasing coupling strength. This result suggests that strong coupling would be preferable in applications where that the transfer function is utilized as a design tool [10].

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